

DISPLACEMENT ENERGY OF UNIT COTANGENT BUNDLES

KEI IRIE

ABSTRACT. For given Riemannian manifold, we study the displacement energy of its unit cotangent bundle in its cotangent bundle. This displacement energy is obviously equal to infinity when the Riemannian manifold is closed. On the otherhand, our main result gives a nice upper bound of this displacement energy when the Riemannian manifold is noncompact. As an application, we prove the existence of a "short" periodic billiard trajectory on any compact Riemannian manifold with boundary.

1. INTRODUCTION

1.1. Main result. First we recall the definition of displacement energy. Let (X, ω) be a symplectic manifold, and K be a compact set on X . Then, the displacement energy of K in X is defined as

$$\inf\{\|H\| \mid H \text{ displaces } K\},$$

where $H = (H_t)_{0 \leq t \leq 1}$ is a time-dependent smooth Hamiltonian with compact support, and $\|H\| := \int_0^1 \sup H_t - \inf H_t dt$.

Let M be a n -dimensional Riemannian manifold without boundary. DT^*M denotes the unit cotangent bundle on M , i.e.

$$DT^*M := \{(q, p) \in T^*M \mid |p| < 1\}.$$

In this paper, we study the following quantity:

$$d(M) := \sup_K \text{displacement energy of } K \text{ in } T^*M,$$

where K runs over all compact sets in DT^*M .

When M is compact, $d(M) = \infty$ since the zero-section of T^*M is not displaceable by Hamiltonian diffeomorphisms. On the otherhand, our main result gives an upper bound of $d(M)$ when M is noncompact.

Suppose that M is noncompact. For any compact set $K \subset M$, let us define

$$r_M(K) := \sup_{x \in M} \text{dist}_M(x, M \setminus K).$$

Moreover, we define

$$r(M) := \sup_{K \subset M} r_M(K).$$

When M is an open set in \mathbb{R}^n , $r(M)$ is equal to the supremum of radius of balls in M .

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The main theorem of this paper is the following:

Theorem 1.1. *Let n be a positive integer, and M be a n -dimensional non-compact Riemannian manifold without boundary. Then $d(M) \leq \text{const}_n r(M)$.*

Remark 1.2. The last inequality means that: there exists a positive constant c depending only on n , which satisfies $d(M) \leq cr(M)$ for all M .

In [7], C.Viterbo proves the following result, and apply it to prove the existence of a short periodic billiard trajectory (theorem 4.1 in [7]).

Proposition 1.3. *If U is a bounded open set in \mathbb{R}^n , then the displacement energy of $DT^*U := \{(q, p) \mid q \in U, |p| < 1\}$ in $T^*\mathbb{R}^n$ is less than $\text{const}_n \text{vol}(U)^{1/n}$.*

This result easily follows from theorem 1.1.

1.2. Application:periodic billiard trajectory. First we clarify the definition of periodic billiard trajectory.

Definition 1.4. Let M be a Riemannian manifold, possibly with boundary. Then, *periodic billiard trajectory* on M is a continuous map $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow M$, such that there exists a finite set $B_\gamma \subset \mathbb{R}/\mathbb{Z}$ with the following properties:

- (1) On $(\mathbb{R}/\mathbb{Z}) \setminus B_\gamma$, γ is smooth and satisfies $\ddot{\gamma} \equiv 0$.
- (2) For any $t \in B_\gamma$, $\gamma(t) \in \partial M$. Moreover, $\dot{\gamma}_\pm(t) := \lim_{h \rightarrow \pm 0} \dot{\gamma}(t+h)$ satisfy

$$\dot{\gamma}_+(t) + \dot{\gamma}_-(t) \in T_{\gamma(t)}\partial M, \quad \dot{\gamma}_-(t) - \dot{\gamma}_+(t) \in (T_{\gamma(t)}\partial M)^\perp \setminus \{0\}.$$

B_γ is called the set of *bounce times*.

Remark 1.5. In the above definition, a closed geodesic is a periodic billiard trajectory (the set of bounce times is empty).

Proposition 1.6. *Let M be a n -dimensional compact Riemannian manifold with nonempty boundary. Then, there exists a periodic billiard trajectory on M with at most $n+1$ bounce times and its length is less than or equal to $d(\text{int}M)$.*

Proposition 1.6 is observed in [7] (proof of theorem 4.1), although it does not contain the estimate of the number of bounce times. A rigorous proof of proposition 1.6 can be found in [1], which is based on a version of energy-capacity inequality ([6], [3]), and the approximation technique due to [2].

Remark 1.7. Although [1] is working on domains in the Euclidean spaces, their proof is valid for general Riemannian manifolds.

By proposition 1.6, theorem 1.1 implies the following corollary:

Corollary 1.8. *Let M be a n -dimensional compact Riemannian manifold with nonempty boundary. Then, there exists a periodic billiard trajectory on M with at most $n+1$ bounce times and its length is less than or equal to $\text{const}_n r(\text{int}M)$.*

In [4], the above corollary is proved (by quite different methods) when M is a domain in the Euclidean space.

1.3. Organization of the paper. The following of the paper is devoted to the proof of theorem 1.1.

In section 2, we introduce the notion of *width* of Riemannian manifolds (denoted by w), and prove an inequality $d(M) \leq 2w(M)$ (lemma 2.2). Hence theorem 1.1 follows from an inequality $w(M) \leq \text{const}_n r(M)$, and we reduce it to a result on closed Riemannian manifolds (theorem 2.3). The rest of the paper is devoted to the proof of theorem 2.3.

To prove theorem 2.3, first we prove the existence of a thick triangulation of a given closed Riemannian manifold. The rigorous statement of this result is given in section 3 (lemma 3.3). Since the proof of lemma 3.3 is little long and it is not the main interest of this paper, the proof is postponed until the last section 6. The main part of the proof of theorem 2.3 is carried out in section 4 and 5.

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2. WIDTH OF RIEMANNIAN MANIFOLDS

In this section, we introduce the notion of *width* of Riemannian manifolds. Let M be a n -dimensional Riemannian manifold without boundary, and K be a compact set on M . Let us define

$$w_M(K) := \inf \{ \|h\| \mid h \in C_0^\infty(M), |dh| \geq 1 \text{ on } K \},$$

where $\|h\| := \sup h - \inf h$. Moreover, let us define the *width* of M by

$$w(M) := \sup_{K \subset M} w_M(K).$$

In particular, when M is closed, $w(M) = \infty$.

Lemma 2.1. *Let M be a manifold without boundary, and g, g' be Riemannian metrics on M , such that $|\xi|_g \leq c|\xi|_{g'}$ for any $\xi \in TM$. Then $w(M, g) \leq cw(M, g')$.*

Proof. We may assume that $w(M, g') < \infty$. Let K be a compact set on M . For any $\delta > 0$, there exists $h \in C_0^\infty(M)$, such that $|dh|_{g'} \geq 1$ on K and $\|h\| \leq w_{(M, g')}(K) + \delta$. Since $|d(ch)|_g \geq 1$ on K , $w_{(M, g)}(K) \leq c(w_{(M, g')}(K) + \delta)$. Since this holds for any $\delta > 0$, $w_{(M, g)}(K) \leq cw_{(M, g')}(K)$. Hence $w(M, g) \leq cw(M, g')$. \square

The following simple observation is the first key step in the proof of theorem 1.1:

Lemma 2.2. *Let M be a Riemannian manifold without boundary. Then, $d(M) \leq 2w(M)$.*

Proof. We may assume $w(M) < \infty$. Let K be a compact set in DT^*M . For any $\delta > 0$, there exists $h \in C_0^\infty(M)$ such that $\|h\| \leq w(M) + \delta$ and $|dh| \geq 1$ on $\pi_M(K)$, where π_M is the natural projection $T^*M \rightarrow M$. Let $H := h \circ \pi_M \in C^\infty(T^*M)$. Let (q_1, \dots, q_n) be a local chart on M , and let (p_1, \dots, p_n) be the associated chart on cotangent fibers. Then, the Hamiltonian vector field X_H of H is calculated as:

$$X_H = \sum_i \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} = \sum_i \frac{\partial h}{\partial q_i} \frac{\partial}{\partial p_i} = dh.$$

Since $|dh| \geq 1$ on $\pi_M(K)$, and $K \subset DT^*M$, $2H$ displaces K . Although $2H$ is not compactly supported, the 1-parameter group $(\varphi_{2H}^t)_{t \in \mathbb{R}}$ of X_{2H} is well-defined, and $\bigcup_{0 \leq t \leq 1} \varphi_{2H}^t(K)$ is compact. Hence displacement energy of K in T^*M is bounded by $\|2H\| = \|2h\|$, therefore by $2w(M)$. \square

We show that theorem 1.1 is reduced to the following theorem:

Theorem 2.3. *Any closed, connected n -dimensional Riemannian manifold N and a compact set $K \subsetneq N$ satisfy $w_N(K) \leq \text{const}_n \text{diam}(N)$.*

Proof of theorem 1.1 modulo theorem 2.3. Let M be a n -dimensional non-compact Riemannian manifold without boundary, as in theorem 1.1. By lemma 2.2, it is enough to show $w(M) \leq \text{const}_n r(M)$, i.e. any compact set K on M satisfies $w_M(K) \leq \text{const}_n r(M)$. We may assume that M is connected (the general case follows at once from this case). The key step in the proof is the following claim:

For any $\varepsilon > 0$, there exists an open neighborhood W of K in M , a closed, connected Riemannian manifold N , and an isometric embedding $i : W \rightarrow N$ such that $\text{diam}(N) \leq (2 + \varepsilon)r(M)$.

Once the above claim is established, then we can complete the proof as follows:

$$w_M(K) = w_N(i(K)) \leq \text{const}_n \text{diam}(N) \leq \text{const}_n r(M).$$

We prove the claim. It is easy to show that there exists U , a connected open neighborhood of K such that \bar{U} is compact and ∂U is smooth. Take an embedding $E : \partial U \times [-1, 1] \rightarrow \bar{U}$ such that $E(u, 1) = u$, $\text{Im } E \cap K = \emptyset$.

We define a closed manifold N by $N := \bar{U} \times \{0\} \bigcup_{\varphi} \bar{U} \times \{1\}$, where φ is defined by

$$\varphi : \text{Im } E \times \{0\} \rightarrow \text{Im } E \times \{1\} : (E(u, t), 0) \mapsto (E(u, -t), 1).$$

Then, N is connected. Moreover we define an embedding $j_0, j_1 : U \rightarrow N$ by

$$j_k : U \cong U \times \{k\} \hookrightarrow N \quad (k = 0, 1).$$

For any $\varepsilon > 0$, there exists a Riemannian metric g_N on N such that:

- (1) $j_0^* g_N \leq g_M|_U$, where g_M denotes the Riemannian metric on M .
- (2) $\text{diam}(U, j_1^* g_N) \leq \varepsilon r(M)$.
- (3) $j_0^* g_N \equiv g_M$ on some open neighborhood W of K .

Since $\sup_{x \in M} \text{dist}_M(x, M \setminus U) \leq r(M)$, (1) implies $\sup_{x \in N} \text{dist}_N(x, U \times \{1\}) \leq r(M)$. Then (2) implies $\text{diam}(N) \leq (2 + \varepsilon)r(M)$. Finally (3) implies that $i := j_0|_W : W \rightarrow N$ is an isometric embedding. \square

3. THICK TRIANGULATION: STATEMENT

The proof of theorem 2.3 is based on the existence of a "thick" triangulation of a given closed Riemannian manifold. The goal of this section is to state this result (lemma 3.3) in rigorous terms. The proof is postponed until the last section.

We start with a review of the notion of a triangulation. A simplicial complex X is a pair $(V(X), \Sigma(X))$ where $V(X)$ is a set, $\Sigma(X) \subset \{\text{finite subsets of } V(X)\} \setminus \{\emptyset\}$, such that

- For any $v \in V(X)$, $\{v\} \in \Sigma(X)$.
- $\tau \subset \sigma, \sigma \in \Sigma(X) \implies \tau \in \Sigma(X)$.

For each $\sigma \in \Sigma(X)$, we define $\dim \sigma$ as $\dim \sigma := \#\sigma - 1$. For each integer $k \geq 0$, we define $\Sigma_k(X) := \{\sigma \in \Sigma(X) \mid \dim \sigma = k\}$. Moreover, we define a symplcial complex X_k by $V(X_k) = V(X)$, $\Sigma(X_k) := \bigcup_{0 \leq l \leq k} \Sigma_l(X)$. For each $v \in V(X)$, we define $N_T(v) \subset V(X)$ as

$$N_T(v) := \{w \in V(X) \mid \{v, w\} \in \Sigma_1(X)\}.$$

An element of $\Sigma(X)$ is called a *simplex* of X . $V(X), \Sigma(X)$ are sometimes abbreviated as V, Σ .

For any simplex $\sigma = \{v_0, \dots, v_k\}$, we define $|\sigma|, \text{int}|\sigma| \subset \mathbb{R}[V] := \bigoplus_{v \in V} \mathbb{R} \cdot v$ by

$$|\sigma| := \left\{ \sum_{0 \leq j \leq k} t_j v_j \mid 0 \leq t_j \leq 1, \sum_j t_j = 1 \right\}, \quad \text{int}|\sigma| := \left\{ \sum_{0 \leq j \leq k} t_j v_j \mid 0 < t_j \leq 1, \sum_j t_j = 1 \right\}.$$

Moreover, we define $|X| := \bigcup_{\sigma \in \Sigma} |\sigma| \subset \mathbb{R}[V]$.

We equip X with a restriction of the standard metric on $\mathbb{R}[V]$, and we call this metric the standard metric on $|X|$. For any $x \in |X|$, we define $\text{St}_X(x) \subset |X|$ by $\text{St}_X(x) := \bigcup_{x \in |\sigma|} \text{int}|\sigma|$, and $\overline{\text{St}}_X(x)$ denotes its closure in $|X|$.

We introduce some terminologies, following [5] section 8.

Definition 3.1. Let X be a simplicial complex, N be a manifold, and $F : |X| \rightarrow N$.

- (1) F is of C^r -class if and only if $F|_{|\sigma|}$ is of C^r -class for any $\sigma \in \Sigma(X)$.
- (2) When F is of C^1 -class, F is *nondegenerate* if and only if for any $\sigma \in \Sigma(X)$ satisfying $\dim \sigma \geq 1$, $d(F|_{|\sigma|})$ has rank equal to $\dim \sigma$ everywhere on $|\sigma|$.
- (3) When F is of C^1 -class, $(dF)_x : \overline{\text{St}}_X(x) \rightarrow T_{F(x)}N$ is defined for each $x \in |X|$ by

$$(dF)_x(y) := (dF|_{|\sigma|})_x(y - x),$$

where $\sigma \in \Sigma(X)$ such that $x \in |\sigma|$, and y is an arbitrary point on $|\sigma|$.

- (4) F is a C^r -immersion if and only if it is of C^r -class and $(dF)_x : \overline{\text{St}}_X(x) \rightarrow T_{F(x)}N$ is injective for any $x \in |X|$.

- (5) (X, F) is a C^r -triangulation of N , if and only if F is a C^r -immersion and a homeomorphism.

Remark 3.2. Assume that $F : |X| \rightarrow N$ is of C^1 -class. It is known that (X, F) is a triangulation if and only if F is nondegenerate and homeomorphism (theorem 8.4 in [5]).

Finally, we state lemma 3.3.

Lemma 3.3. *Let N be a n -dimensional closed Riemannian manifold. For sufficiently small $\varepsilon > 0$, there exists a C^∞ triangulation (X, F) of N with the following properties, where c_0, c_1, c_2 are positive constants depending only on n .*

- (1) *For any $\sigma \in \Sigma_1(X)$, $F|_{|\sigma|} : |\sigma| \rightarrow N$ is a geodesic.*
- (2) *For any $\sigma \in \Sigma(X)$ and non-zero tangent vector ξ on $|\sigma|$, $|(dF|_{|\sigma|})(\xi)|_N / |\xi| \in [c_0\varepsilon, c_1\varepsilon]$, where $|\xi|$ is defined by the standard metric on $|X|$.*
- (3) *For any $u \in V(X)$, $\sharp N_X(u) \leq c_2$.*

Remark 3.4. It is easy to see that (3) follows from (2).

We fix the above constants c_0, c_1, c_2 until the end of this paper.

4. WIDTH OF A MANIFOLD WITH MANY HOLES

The main result of this section is the following proposition:

Proposition 4.1. *Let N be a n -dimensional closed Riemannian manifold, and ε be a sufficiently small positive number so that there exists a triangulation (X, F) of N as in lemma 3.3. Then,*

$$w(N \setminus F(V(X))) \leq \text{const}_n \varepsilon.$$

$N \setminus F(V(X))$ is, roughly speaking, a manifold with many "holes". proposition 4.1 claims that width of such a manifold is sufficiently small. The following of this section is devoted to the proof of proposition 4.1.

By (3) in lemma 3.3, there exists a map $h : V(X) \rightarrow \{j \in \mathbb{Z} \mid 0 \leq j \leq c_2\}$ such that for any symplex $\sigma = \{v_0, \dots, v_k\}$ of X , $h(v_0), \dots, h(v_k)$ are distinct.

Then we extend h to a continuous function on $|X|$ (still denoted by h) as follows: for any symplex $\sigma = \{v_0, \dots, v_k\}$ of X , $h|_{|\sigma|}$ is defined by

$$h\left(\sum_{0 \leq j \leq k} t_j v_j\right) := \sum_{0 \leq j \leq k} t_j h(v_j).$$

We define a continuous function h' on N by $h' := h \circ F^{-1}$. Obviously $0 \leq h' \leq c_2$. Although h' is not of C^∞ , it is of C^∞ on $U := \bigcup_{\sigma \in \Sigma_n(X)} F(\text{int}|\sigma|)$.

Let ρ be a $\mathbb{R}_{\geq 0}$ valued smooth function defined on $\mathbb{R}_{\geq 0}$, such that ρ is constant near 0, $\text{supp} \rho \subset [0, 1]$ and

$$\int_{\mathbb{R}^n} \rho(|x|) dx = 1.$$

For $\delta > 0$, let us define $\rho_\delta \in C_0^\infty(\mathbb{R}_{\geq 0})$ by $\rho_\delta(t) := \delta^{-n} \rho(t/\delta)$. When $0 < \delta < \text{inj}(N) :=$ the injectivity radius of N , the following formula defines a C^∞ function h_δ on N :

$$h_\delta(x) := \int_{T_x N} h'(\exp_x(\zeta)) \rho_\delta(|\zeta|) d\text{vol}_x(\zeta).$$

\exp_x denotes the exponential map at x , and vol_x denotes the volume form on $T_x N$ defined by the Riemannian metric on N . $0 \leq h' \leq c_2$ implies that $\|h_\delta\| \leq c_2$. We prove the following lemma:

Lemma 4.2. *For any compact set $K \subset N \setminus F(V(X))$, $\liminf_{\delta \rightarrow 0} \inf_K |dh_\delta| \geq \text{const}_n \varepsilon^{-1}$.*

First we point out that proposition 4.1 follows from lemma 4.2. Denote the constant in lemma 4.2 by c . Then, for any $c' \in (0, c)$, $\inf_K |dh_\delta| \geq c' \varepsilon^{-1}$ for sufficiently small δ . Setting $h := c'^{-1} \varepsilon h_\delta$, h satisfies $\|h\| \leq c'^{-1} c_2 \varepsilon$ and $\inf_K |dh| \geq 1$. Hence

$$w(N \setminus F(V(X))) = \sup_K w_{N \setminus F(V(X))}(K) \leq \text{const}_n \varepsilon.$$

To prove lemma 4.2, first we need the following sublemma.

Lemma 4.3. *For any $x \in N \setminus F(V(X))$, there exists $\xi \in T_x N$ with the following property: if $\zeta \in TU \subset TN$ is sufficiently close to ξ in TN , then $dh'(\zeta)/|\zeta| \geq c\varepsilon^{-1}$, where c is a positive constant depending only on n .*

Proof. Let $\sigma = \{v_0, \dots, v_k\}$ be the unique simplex of X such that $F^{-1}(x) \in \text{int}|\sigma|$. Since $x \notin F(V(X))$, $k \geq 1$. Since $h(v_0), \dots, h(v_k)$ are distinct integers and $\text{diam}|\sigma| \leq \sqrt{2}$, there exists $\eta \in T_{F^{-1}(x)}|\sigma|$ such that $dh(\eta)/|\eta| \geq 1/\sqrt{2}$. Setting $\xi := dF(\eta)$, (2) in lemma 3.3 implies that $dh'(\xi)/|\xi| \geq (\sqrt{2}c_1\varepsilon)^{-1}$. Then ξ satisfies the requirement of the lemma with any $c < (\sqrt{2}c_1)^{-1}$. \square

Finally we prove lemma 4.2. First we introduce the following notation:

$$B_N(x, r) := \{y \in N \mid \text{dist}_N(x, y) \leq r\} \quad (x \in N, r \in \mathbb{R}_{\geq 0}).$$

Proof of lemma 4.2. Let us define a map e by

$$e : \{(x, \xi) \in TN \mid |\xi| \leq \text{inj}(N)\} \rightarrow N; \quad (x, \xi) \mapsto \exp_x(\xi).$$

For any $\xi \in T_x N$, let $\tilde{\xi}$ be its horizontal lift (with respect to the Levi-Civita connection). Then,

$$dh_\delta(\xi) = \int_{T_x N} dh'(de(\tilde{\xi}(\zeta))) \rho_\delta(|\zeta|) d\text{vol}_x(\zeta).$$

Since h' is smooth on U (hence, almost everywhere on N), the right hand side makes sense. For each $x \in K$, take a vector field ξ defined near x , so that $\xi(x)$ satisfies the requirement of the lemma 4.3. Since $de(\tilde{\xi}(x, 0)) = \xi(x)$, the following inequality holds for sufficiently small $r, \delta > 0$:

$$dh'(de(\tilde{\xi}(\zeta)))/|de(\tilde{\xi}(\zeta))| \geq c\varepsilon^{-1} \quad (y \in B_N(x, r), \zeta \in T_y N, |\zeta| \leq \delta),$$

Moreover, by taking r, δ sufficiently small, we may also assume that the following holds:

$$\begin{aligned} |de(\tilde{\xi}(\zeta))|/|\xi(x)| &\geq 1/2 & (y \in B_N(x, r), \zeta \in T_y N, |\zeta| \leq \delta), \\ |\xi(z)|/|\xi(x)| &\leq 2 & (z \in B_N(x, r)). \end{aligned}$$

We denote ξ, r, δ by ξ_x, r_x, δ_x .

Since K is compact, there exist finitely many points $x_1, \dots, x_m \in K$ such that $\{B_N(x_i, r_{x_i})\}_{1 \leq i \leq m}$ covers K . Let $\delta := \min_{1 \leq i \leq m} \delta_{x_i}$. For any $y \in K$, there exists $1 \leq i \leq m$ such that $y \in B_N(x_i, r_{x_i})$. Then, we get

$$\begin{aligned} dh_\delta(\xi_{x_i}(y)) &= \int_{T_y N} dh'(de(\tilde{\xi}_{x_i}(\zeta))) \rho_\delta(|\zeta|) d\text{vol}_y(\zeta) \\ &\geq c\varepsilon^{-1} \int_{T_y N} |de(\tilde{\xi}_{x_i}(\zeta))| \rho_\delta(|\zeta|) d\text{vol}_y(\zeta) \\ &\geq c(4\varepsilon)^{-1} |\xi_{x_i}(y)|. \end{aligned}$$

Hence $|dh_\delta(y)| \geq c(4\varepsilon)^{-1}$ for any $y \in K$. □

5. PROOF OF THEOREM 2.3

The main result of this section is the following proposition:

Proposition 5.1. *Let N be a closed, connected n -dimensional Riemannian manifold, (X, F) be a triangulation of N which satisfies the requirements of lemma 3.3 for $\varepsilon > 0$. Then, the following holds for sufficiently small ε :*

For any $v \in V(X)$, there exists an open set $W \subset N \setminus F(V(X))$ and a diffeomorphism $\Phi : W \rightarrow N \setminus \{F(v)\}$ such that

$$|d\Phi(\xi)|/|\xi| \leq \text{const}_n \text{diam}(N) \varepsilon^{-1}$$

for any non-zero tangent vector ξ on W .

First we point out that theorem 2.3 follows from proposition 5.1. Let N be a closed, connected n -dimensional Riemannian manifold, and $K \subsetneq N$ be a compact set on N . We have to show that $w_N(K) \leq \text{const}_n \text{diam}(N)$.

Take $\varepsilon > 0$ as in proposition 5.1, and (X, F) be a triangulation which satisfies requirements in lemma 3.3 for ε . We may assume that $F(V(X)) \setminus K \neq \emptyset$.

Take an arbitrary $v \in V(X) \setminus F^{-1}(K)$, and take W, Φ as in the claim in proposition 5.1. Then

$$\begin{aligned} w(N \setminus \{F(v)\}) &\leq \text{const}_n \text{diam}(N) \varepsilon^{-1} w(W) \\ &\leq \text{const}_n \text{diam}(N) \varepsilon^{-1} w(N \setminus F(V(X))) \\ &\leq \text{const}_n \text{diam}(N). \end{aligned}$$

The first inequality follows from lemma 2.1, and the last inequality follows from proposition 4.1. Since $w_N(K) = w_{N \setminus \{F(v)\}}(K) \leq w(N \setminus \{F(v)\})$, it completes the proof of theorem 2.3.

5.1. Sketch of the proof of proposition 5.1. We sketch the proof of proposition 5.1. Details are carried out in section 5.2 - 5.4.

Let $N, \varepsilon, (X, F)$ are as in proposition 5.1. g_N denotes the Riemannian metric on N . First we show the following lemma, which is proved in section 5.2.

Lemma 5.2. *There exists a tree $T \subset X$ such that $V(T) = V(X)$ and*

$$\text{diam}(|T|, F^*g_N|_{|T|}) \leq \text{const}_n \text{diam}(N).$$

Remark 5.3. *Tree* means a simply-connected 1-dimensional symplectic complex.

Let $T \subset X$ be a tree as in lemma 5.2. In the following, we equip $|T|$ with the metric $F^*g_N|_{|T|}$ unless otherwise stated. In section 5.3, we define a Riemannian manifold (with boundary) \tilde{T} , and an embedding map $i : |T| \rightarrow \tilde{T}$. Roughly speaking, \tilde{T} is obtained by "fatten" $|T|$. Moreover, we show that \tilde{T} and i satisfy the following properties: (lemma 5.4 is proved in section 5.3, and lemma 5.5 is proved in section 5.4.)

Lemma 5.4. *For any $0 < \delta < 1$, there exists $\varepsilon(\delta, N) > 0$ such that the following holds: If $0 < \varepsilon < \varepsilon(\delta, N)$, there exists an embedding $I : \tilde{T} \rightarrow N$ such that:*

- (1) $I \circ i : |T| \rightarrow N$ is equal to $F|_{|T|}$.
- (2) Any nonzero tangent vector ξ on \tilde{T} satisfies $|dI(\xi)|_N / |\xi|_{\tilde{T}} \in [1 - \delta, 1 + \delta]$.

Lemma 5.5. *For any $v \in V(T)$, there exists a neighborhood Z of $\partial\tilde{T}$ in \tilde{T} , and a diffeomorphism $\varphi : Z \rightarrow \tilde{T} \setminus \{i(v)\}$ with the following properties:*

- (1) $\varphi \equiv \text{id}$ on some neighborhood of $\partial\tilde{T}$.
- (2) Any nonzero tangent vector ξ on Z satisfies $|d\varphi(\xi)|_{\tilde{T}} / |\xi|_{\tilde{T}} \leq \text{const}_n \text{diam}(|T|)\varepsilon^{-1}$.

We prove proposition 5.1 assuming those results. Suppose that $0 < \varepsilon < \varepsilon(1/2, N)$, and take $I : \tilde{T} \rightarrow N$ as in lemma 5.4. For each $v \in V(X)$, define $W \subset N$ by $W := (N \setminus I(\tilde{T})) \cup I(Z)$, and define $\Phi : W \rightarrow N \setminus \{v\}$ by

$$\Phi(x) = \begin{cases} x & (x \in N \setminus I(\tilde{T})) \\ I \circ \varphi \circ I^{-1}(x) & (x \in I(Z)) \end{cases}.$$

We check that any nonzero tangent vector $\xi \in T_x W$ satisfies $|d\Phi(\xi)| / |\xi| \leq \text{const}_n \text{diam}(N)\varepsilon^{-1}$. If $x \in I(Z)$, then

$$|d\Phi(\xi)| / |\xi| \leq \text{const}_n \text{diam}(|T|)\varepsilon^{-1} \leq \text{const}_n \text{diam}(N)\varepsilon^{-1}.$$

The first inequality follows from lemma 5.4 (2) and lemma 5.5 (2), the second inequality follows from lemma 5.2. On the otherhand, if $x \notin I(\tilde{T})$, $d\Phi(\xi) = \xi$.

5.2. Proof of lemma 5.2. Since (X, F) satisfies lemma 3.3 (2), lemma 5.2 follows from the following lemma:

Lemma 5.6. *Let X be a symplectic complex such that $|X|$ is connected. Then, there exists a tree $T \subset X$ such that $V(T) = V(X)$ and $\text{diam}(|T|) \leq \text{const}_n \text{diam}(|X|)$, where $\text{diam}(|T|), \text{diam}(|X|)$ are defined with respect to the standard metrics (see section 3).*

Proof. First we show the following claim:

For any $k \in \{2, \dots, n\}$, $\text{diam}(|X_{k-1}|) \leq \text{const}_k \cdot \text{diam}(|X_k|)$.

Let $c : [0, 1] \rightarrow |X_k|$ be a piecewise linear map such that $c(0), c(1) \in |X_{k-1}|$. Then there exists $0 = t_0 < t_1 < \dots < t_m = 1$ such that:

- $c(t_0), \dots, c(t_m) \in |X_{k-1}|$.
- For any $i = 1, \dots, m$ there exists $\sigma_i \in \Sigma_k(X)$ such that $c([t_{i-1}, t_i]) \subset |\sigma_i|$.

For each $i = 1, \dots, m$, there exists $c_i : [t_{i-1}, t_i] \rightarrow |\partial\sigma_i|$ such that $c_i(t_{i-1}) = c(t_{i-1})$, $c_i(t_i) = c(t_i)$ and $l(c_i) \leq \text{const}_k \cdot l(c|_{[t_{i-1}, t_i]})$ (l denotes the lengths of curves). By connecting c_1, \dots, c_m we get a map $c' : [0, 1] \rightarrow |X_{k-1}|$ such that $c'(0) = c(0)$, $c'(1) = c(1)$, $l(c') \leq \text{const}_k \cdot l(c)$. Hence we have proved the above claim. By applying the above claim for $k = 2, \dots, n$, we get $\text{diam}(|X_1|) \leq \text{const}_n \cdot \text{diam}(|X|)$.

Take an arbitrary function $\rho : \Sigma_1(X) \rightarrow [1, 2]$ such that $\{\rho(\sigma)\}_{\sigma \in \Sigma_1(X)}$ are linearly independent over \mathbb{Q} . A *path on X* means a subcomplex of X which is isomorphic (as a symplcial complex) to some P_l ($l = 1, 2, \dots$), where P_l is a 1-dimensional symplcial complex defined as

$$V(P_l) = \{0, \dots, l\}, \quad \Sigma_1(P_l) = \{\{0, 1\}, \{1, 2\}, \dots, \{l-1, l\}\}.$$

For any path γ on X , let $\rho(\gamma) := \sum_{\sigma \in \Sigma_1(\gamma)} \rho(\sigma)$. If two paths γ, γ' satisfy $\rho(\gamma) = \rho(\gamma')$, then $\gamma = \gamma'$.

Fix an arbitrary element $v_0 \in V$. For each $v \in V$, let γ_v be the path on X connecting v and v_0 , which attains the minimum value of ρ . Then $\rho(\gamma_v) \leq \text{const}_n \cdot \text{diam}(|X|)$, since $\text{diam}(|X_1|) \leq \text{const}_n \cdot \text{diam}(|X|)$ and $\rho(\sigma) \leq 2$ for any $\sigma \in \Sigma_1(X)$.

Let T be the union of γ_v , where v runs over all elements of V . Then, it is easy to check that T is a tree. Moreover, $\text{diam}(|T|) \leq \text{const}_n \cdot \text{diam}(|X|)$, since for any $v, w \in V$

$$\text{dist}_{|T|}(v, w) \leq \text{dist}_{|T|}(v, v_0) + \text{dist}_{|T|}(v_0, w) \leq \text{const}_n \cdot \text{diam}(|X|).$$

□

5.3. Definition of \tilde{T} and i . Let $N, \varepsilon, (X, F)$ are as in proposition 5.1, and $T \subset X$ be a tree as in lemma 5.2. First we define a Riemannian manifold $T(r)$ and an embedding $i_r : |T| \rightarrow T(r)$ for sufficiently small $r > 0$.

To spell out the definition, we introduce some notations. For $p, q \in N$ such that $\text{dist}(p, q) \leq \text{inj}(N)$, we define $\vec{pq} \in T_p N$ by

$$\vec{pq} := \exp_p^{-1}(q).$$

Moreover, when $p \neq q$, $e_{pq} := \vec{pq}/|\vec{pq}|$, and $H_{pq} \subset T_p N$ denotes the orthogonal complement of $\vec{pq} \cdot \mathbb{R}$.

We start to define $T(r)$ and i_r . Let $\mu : [0, \infty) \rightarrow [0, \infty)$ be a continuous map satisfying the following properties with respect to $c > 0$:

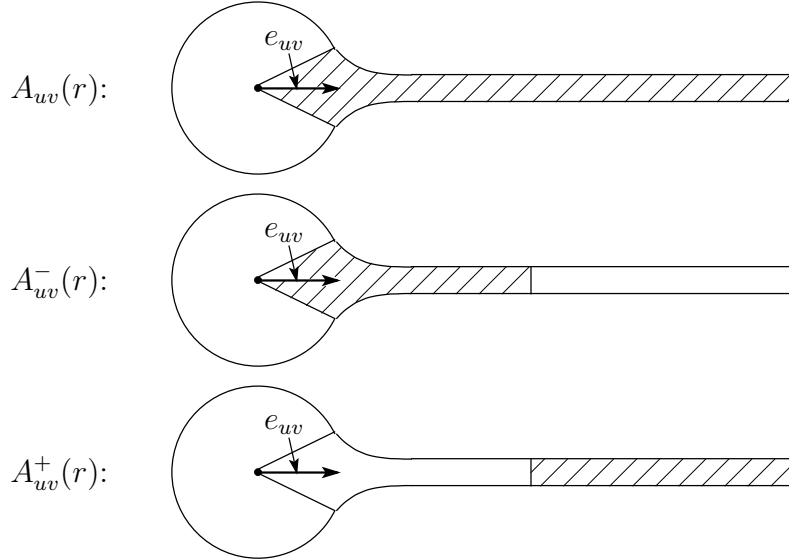
- $\mu(t) = ct$ for $0 \leq t \leq 1/\sqrt{1+c^2}$.
- $\mu(t) = \sqrt{1-t^2}$ when $t \geq 1/\sqrt{1+c^2}$ and t is sufficiently close to $1/\sqrt{1+c^2}$.
- $\mu(t) \equiv c/2\sqrt{1+c^2}$ when $t \geq 1$.
- $\mu(t)$ is a non-increasing function for $t \geq 1/\sqrt{1+c^2}$.

Remark 5.7. In the following, we identify each $u \in V(T)$ with $F(u) \in N$.

Let $u \in V(T)$, $v \in N_T(u)$ and $r > 0$. We define $A_{uv}^-(r), A_{uv}^+(r), A_{uv}(r) \subset T_u N$ as follows (d_{uv} abbreviates $\text{dist}_N(u, v)$):

$$\begin{aligned} A_{uv}(r) &:= \{h + te_{uv} \mid 0 \leq t \leq 2d_{uv}/3, h \in H_{uv}, |h| \leq r\mu(t/r)\}, \\ A_{uv}^-(r) &:= \{h + te_{uv} \mid 0 \leq t \leq d_{uv}/3, h \in H_{uv}, |h| \leq r\mu(t/r)\}, \\ A_{uv}^+(r) &:= \{h + te_{uv} \mid d_{uv}/3 \leq t \leq 2d_{uv}/3, h \in H_{uv}, |h| \leq r\mu(t/r)\}. \end{aligned}$$

We equip $A_{uv}(r), A_{uv}^\pm(r)$ with the metric on $T_u N$.



Since the triangulation (X, F) satisfies lemma 3.3 (2), there exists $\theta_0 > 0$, which depends only on c_0 such that: Any $u \in V(T)$, $v, v' \in N_T(u)$, $v \neq v'$ satisfies $|\angle vuv'| \geq \theta_0$. Hence if c is sufficiently small, the following holds:

$$u \in V(T), \quad v, v' \in N_T(u), \quad v \neq v' \implies A_{uv}(r) \cap A_{uv'}(r) = \{0\}.$$

We fix such $c > 0$ and denote it as c_3 . The constant c_3 is fixed until the end of this section.

Let $i : H_{uv} \rightarrow H_{vu}$ be an isometry defined by a parallel transport along the geodesic segment connecting u and v . When $r < d_{uv}/3$, then $\mu(t/r) \equiv c_3/2\sqrt{1+c_3^2}$ for $d_{uv}/3 \leq t \leq 2d_{uv}/3$. Therefore we can define an isometry $\psi_{uv} : A_{uv}^+(r) \rightarrow A_{vu}^+(r)$ by

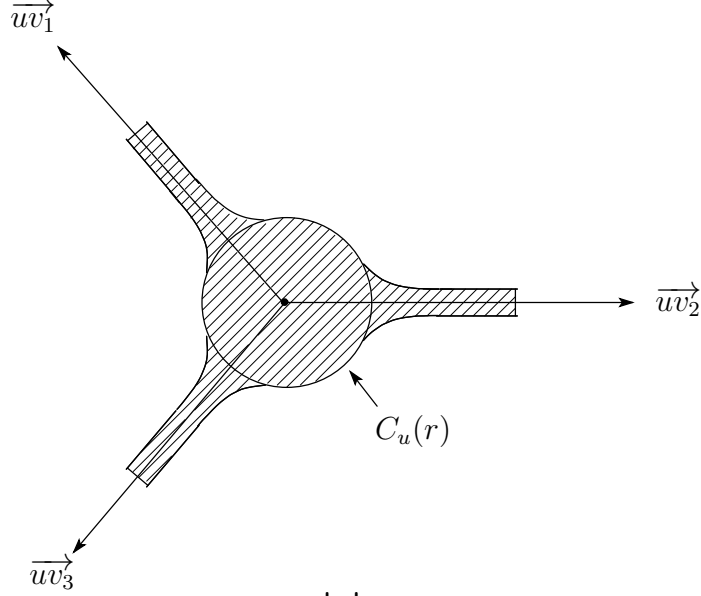
$$\psi_{uv}(h + te_{uv}) = i(h) + (d_{uv} - t)e_{vu}.$$

Let $r > 0$ be a sufficiently small number so that $r < d_{uv}/3$ for any $u \in V(T)$, $v \in N_T(u)$. (This is satisfied when $r/\varepsilon < (\sqrt{2}/3)c_0$, since lemma 3.3 (1), (2) imply $d_{uv} \geq \sqrt{2}c_0\varepsilon$). For

each $u \in V(T)$, let us define

$$B_u(r) := \{x \in T_u N \mid |x| \leq r\},$$

$$C_u(r) := B_u(r) \cup \bigcup_{v \in N_T(u)} A_{uv}(r).$$



We define an equivalence relation \sim on $\bigcup_{u \in V(T)} C_u(r)$ so that: $x \sim y$ if and only if $x = y$ or $x \in A_{uv}^+(r)$, $y \in A_{vu}^+(r)$, $y = \psi_{uv}(x)$ for some $u \in V(T)$, $v \in N_T(u)$.

Then $T(r)$ is defined as $T(r) := \bigcup_{u \in V(T)} C_u(r) / \sim$. Since ψ_{uv} are isometries, $T(r)$ carries a natural Riemannian metric $g_{T(r)}$. $|\cdot|_{g_{T(r)}}$ is abbreviated as $|\cdot|_{T(r)}$.

Moreover, we define $i_r : |T| \rightarrow T(r)$ by

$$i_r((1-t)u + tv) := [t\vec{uv}] \quad (0 \leq t \leq 2/3),$$

where $u \in V(T)$, $v \in N_T(u)$ (since $\psi_{uv}(t\vec{uv}) = (1-t)\vec{vu}$, this is well-defined).

Remark 5.8. The following remarks are immediate consequences of the definition:

- If $r' \leq r$, there exists a natural isometric embedding $T(r') \rightarrow T(r)$. In the following, we consider $T(r')$ as a subset of $T(r)$, using this embedding map.
- If S is a subtree of T , there exists a natural isometric embedding $S(r) \rightarrow T(r)$. In the following, we consider $S(r)$ as a subset of $T(r)$, using this embedding map.

This completes the definitions of $T(r)$ and i_r . Next we define \tilde{T} and $i : |T| \rightarrow \tilde{T}$. First we need the following lemma:

Lemma 5.9. *There exists a positive constant ρ_n depending only on n , which satisfies the following:*

For any $0 < \delta < 1$, there exists $\varepsilon(\delta, N) > 0$ such that: if $0 < \varepsilon < \varepsilon(\delta, N)$ and $r \leq \varepsilon \rho_n$, there exists an embedding $I : T(r) \rightarrow N$ satisfying

- (a): $I \circ i_r : |T| \rightarrow N$ is equal to $F|_{|T|}$.
(b): $|dI(\xi)|_N / |\xi|_{T(r)} \in [1 - \delta, 1 + \delta]$ for any non-zero tangent vector ξ on $T(r)$.

We fix $\rho_n > 0$ as in lemma 5.9, and define \tilde{T} and i as

$$\tilde{T} := T(\varepsilon \rho_n), \quad i := i_{\varepsilon \rho_n}.$$

Then, this definition clearly satisfies lemma 5.4.

To spell out the proof of lemma 5.9, we introduce the following notation: for $p, q \in N$ such that $\text{dist}_N(p, q) \leq \text{inj}(N)$, $\gamma_{pq} : [0, 1] \rightarrow N$ denotes the shortest geodesic segment such that $\gamma_{pq}(0) = p$, $\gamma_{pq}(1) = q$.

Proof of lemma 5.9. Setting cut off function $\chi : [1/3, 2/3] \rightarrow [0, 1]$ such that $\chi \equiv 0$ near $1/3$, $\chi \equiv 1$ near $2/3$ and $\chi(t) + \chi(1 - t) = 1$, we define $I : T(r) \rightarrow N$ as follows:

- (i) For any $u \in V(T)$, $I([x]) := \exp_u(x)$ where $x \in C_u^-(r) := B_u(r) \cup \bigcup_{v \in N_T(u)} A_{uv}^-(r)$.
(ii) For any $u \in V(T)$, $v \in N_T(u)$ and $x = h + t\vec{uv} \in A_{uv}^+(r)$,

$$I([x]) := \gamma_{\exp_u(x) \exp_v(\psi_{uv}(x))}(\chi(t)).$$

Since $\chi(t) + \chi(1 - t) = 1$, this is well-defined.

Since F satisfies lemma 3.3 (1), I satisfies (a). If the metric of N is flat on some neighborhood of u , then $I|_{C_u(r)}$ is isometric. Therefore I satisfies (b) for sufficiently small ε . In particular, I is an immersion.

To show that I is an embedding for sufficiently small ε , it is enough to check that I is injective. For each $u \in V(T)$, define a tree $T_u \subset T$ by

$$V(T_u) := \{u\} \cup N_T(u), \quad \Sigma_1(T_u) := \{\{u, v\} \mid v \in N_T(u)\}.$$

We consider $T_u(r)$ as a subset of $T(r)$.

For any $v \in N_T(u)$, $\text{dist}_N(u, v) \geq \sqrt{2}c_0\varepsilon$. Moreover, recall that: any $u \in V(T)$, $v, v' \in N_T(u)$, $v \neq v'$ satisfy $|\angle vuv'| \geq \theta_0$, where θ_0 depends only on c_0 . Hence if r/ε is sufficiently small compared to c_0 , $I|_{T_u(r)}$ is an embedding (hence injective) for sufficiently small ε .

Following claims are easily verified from lemma 3.3 (2):

- If $u \in V(X)$ and $e \in \Sigma_1(X)$ satisfies $u \notin e$, then $\text{dist}_N(u, F(|e|)) \geq c_0\varepsilon$.
- If $e, e' \in \Sigma_1(X)$ are disjoint, then $\text{dist}_N(F(|e|), F(|e'|)) \geq c_0\varepsilon$.

Let $x, y \in T(r)$. Suppose that there exists no $u \in V(T)$ such that $x, y \in T_u(r)$. Then, at least one of the following holds:

- There exist $u, u' \in V(T)$, such that $u \neq u'$ and $x \in B_u(r)$, $x' \in B_{u'}(r)$.
- There exist $u \in V(T)$, $e \in \Sigma_1(T)$, $u \notin e$ and $x \in B_u(r)$, $y \in \bar{e}(r)$ (\bar{e} denotes the subtree of T such that $\Sigma_1(\bar{e}) = \{e\}$).
- There exist $u \in V(T)$, $e \in \Sigma_1(T)$, $u \notin e$ and $x \in \bar{e}(r)$, $y \in B_u(r)$.

- There exist $e, e' \in \Sigma_1(T)$, $e \cap e' = \emptyset$ and $x \in \bar{e}(r)$, $y \in \bar{e}'(r)$.

In all cases, $\text{dist}_N(I(x), I(y)) \geq c_0\varepsilon - 2r$. Therefore, if $r/\varepsilon < c_0/2$, $I(x) \neq I(y)$. \square

5.4. Proof of lemma 5.5. Let us denote $r := \rho_n\varepsilon$, i.e. $\tilde{T} = T(r)$. First we restate what we have to show (we denote v in the statement of lemma 5.5 as v_1):

For any $v_1 \in V(T)$, there exists $Z \subset T(r)$, a neighborhood of $\partial T(r)$, and a diffeomorphism $\varphi : Z \rightarrow T(r) \setminus \{i(v_1)\}$, such that any nonzero tangent vector ξ on Z satisfies $|d\varphi(\xi)|_{T(r)}/|\xi|_{T(r)} \leq \text{const}_n \text{diam}(|T|)\varepsilon^{-1}$.

We use abbreviations $d := \text{diam}(|T|)$ and $d(v) := \text{dist}_{|T|}(v, v_1)$ ($\forall v \in V(T)$).

Remark 5.10. For any $e \in \Sigma_1(T)$, $(\text{length of } F(|e|))/\varepsilon \in [c_0, c_1]$, since F satisfies (2) in lemma 3.3. Therefore, it is enough to prove lemma 5.5 assuming that lengths of $F(|e|)$ are same for all $e \in \Sigma_1(T)$.

For any subtree S of T such that $v_1 \in V(S)$, let ν_S be the inward normal vector of $\partial S(r/2)$ in $S(r)$ (we consider $S(r/2)$ as a submanifold of $S(r)$: recall remark 5.8). For sufficiently small $c > 0$, there exists an embedding $E_S : \partial S(r/2) \times (-cr, cr) \rightarrow S(r)$ such that (t denotes the coordinate on $(-cr, cr)$):

$$E_S(z, 0) = z, \quad \partial_t E_S(z, 0) = \nu_S(z), \quad \partial_t^2 E_S(z, t) = 0.$$

In the last equation, ∂_t^2 is defined by the Levi-Civita connection associated with $g_{S(r)}$. Note that we may take $c > 0$ so that it depends only on n (hence, independent on T and S). We fix such c and denote it by c_4 .

Define a manifold X_S by

$$X_S := (T(r) \setminus S(r/2)) \bigcup_{E_S|_{\partial S(r/2) \times (-c_4r, 0]}} \partial S(r/2) \times (-c_4r, d).$$

We equip X_S with a metric g_{X_S} , which is defined in the following manner. First, we define a metric g on $\partial S(r/2) \times (-c_4r, d)$ by

$$g := (\text{pr}_{\partial S(r/2)})^*(g_{T(r)}|_{\partial S(r/2)}) + dt^2.$$

Setting cut off function $\alpha : (-c_4, 0] \rightarrow [0, 1]$ such that $\alpha \equiv 1$ near $-c_4$ and $\alpha \equiv 0$ near 0, we define a metric g_{X_S} on X_S so that

- $g_{X_S} = g_{T(r)}$ on $(T(r) \setminus S(r/2)) \setminus \text{Im } E_S$.
- $g_{X_S} = \alpha(t/r)E_S^*(g_{T(r)}) + (1 - \alpha(t/r))g$ on $\partial S(r/2) \times (-c_4r, 0]$.
- $g_{X_S} = g$ on $\partial S(r/2) \times [0, d]$.

Consider the case $S = T$. Then, there exists a diffeomorphism

$$\kappa : (T(r) \setminus T(r/2)) \cup \text{Im } E_T \rightarrow X_T$$

such that $|d\kappa(\xi)|_{X_T}/|\xi|_{T(r)} \leq \text{const}_n d\varepsilon^{-1}$ for any nonzero tangent vector ξ . Hence it is enough to show the following lemma:

Lemma 5.11. *There exists $Y \subset X_T$, a neighborhood of $\partial T(r)$ in X_T , and a diffeomorphism $\varphi' : Y \rightarrow T(r) \setminus \{i(v_1)\}$ such that any nonzero tangent vector ξ on Y satisfies $|d\varphi'(\xi)|_{T(r)}/|\xi|_{X_T} \leq \text{const}_n$.*

Actually, once we prove lemma 5.11, $Z := \kappa^{-1}(Y)$ and $\varphi := \varphi' \circ \kappa$ satisfy $|d\varphi(\xi)|_{T(r)}/|\xi|_{T(r)} \leq \text{const}_n d\varepsilon^{-1}$ for any nonzero tangent vector ξ on Z .

To prove lemma 5.11, first we define $Y \subset X_T$. Fix a cut off function $\chi : [1/3, 2/3] \rightarrow [0, 1]$ such that $\chi \equiv 1$ near $1/3$, $\chi \equiv 0$ near $2/3$ and $\chi(t) + \chi(1-t) = 1$. We define $\bar{d} \in C^\infty(T(r/2))$ as follows:

- For each $v \in V(T)$, $\bar{d} \equiv d(v)$ on $C_v^-(r/2) := B_v(r/2) \cup \bigcup_{w \in N_T(v)} A_{vw}^-(r/2)$.
- For each $v \in V(T)$, $w \in N_T(v)$ and $x = h + t\vec{vw} \in A_{vw}^+(r/2)$,
 $\bar{d}([x]) := \chi(t)d(v) + \chi(1-t)d(w)$.

Since $\chi(t) + \chi(1-t) = 1$, this is well-defined.

Next we define $Y_S \subset X_S$ by

$$Y_S := (T(r) \setminus S(r/2)) \cup \{(z, t) \mid z \in \partial S(r/2), 0 \leq t < \bar{d}(z)\}.$$

We equip Y_S with the metric $g_{X_S}|_{Y_S}$, and denote it by g_{Y_S} .

We define $Y \subset X_T$ by $Y := Y_T$. We have to show that there exists a diffeomorphism $\varphi' : Y_T \rightarrow T(r) \setminus \{i(v_1)\}$, such that any non-zero tangent vector ξ on Y satisfies $|d\varphi'(\xi)|_{T(r)}/|\xi|_{Y_T} \leq \text{const}_n$.

Let $m := |V(T)|$, and take an arbitrary increasing sequence of subtrees of T :

$$\{v_1\} = T_1 \subset T_2 \subset \dots \subset T_m = T.$$

In the following, we abbreviate $|\cdot|_{Y_{T_j}}$ by $|\cdot|_j$.

To spell out the proof, we introduce some notations:

- For any $u \in V(T)$, $v \in N_T(u)$ and $0 \leq a \leq r$, we define $R_{uv}^0(a), R_{uv}^1(a) \subset B_u(r)$ by

$$R_{uv}^0(a) := \{h + te_{uv} \mid \sqrt{|h|^2 + t^2} = a, |h| \leq c_3 t\},$$

$$R_{uv}^1(a) := \{h + te_{uv} \mid \sqrt{|h|^2 + t^2} = a, |h| \geq c_3 t\}.$$

Moreover, for $0 \leq b < c \leq r$, we define $R_{uv}^0(b, c), R_{uv}^1(b, c) \subset B_u(r)$ by

$$R_{uv}^0(b, c) := \{h + te_{uv} \mid b < \sqrt{|h|^2 + t^2} < c, |h| \leq c_3 t\},$$

$$R_{uv}^1(b, c) := \{h + te_{uv} \mid b < \sqrt{|h|^2 + t^2} < c, |h| \geq c_3 t\}.$$

$R_{uv}^i[b, c], R_{uv}^i(b, c], R_{uv}^i[b, c]$ ($i = 0, 1$) are defined in similar way.

Remark 5.12. Recall that we choose c_3 so that the following holds:

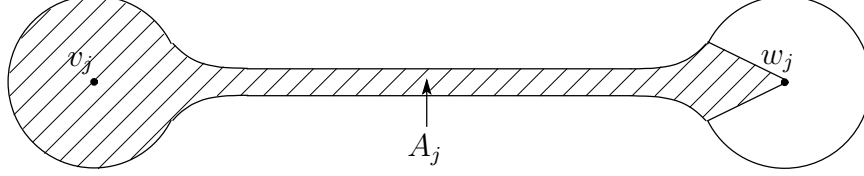
$$u \in V(T), \quad v, v' \in N_T(u), \quad v \neq v' \implies A_{uv}(r) \cap A_{uv'}(r) = \{0\}.$$

Hence, the following holds for any $u \in V(T)$:

$$v, v' \in N_T(u), \quad v \neq v' \implies R_{uv}^0(a) \subset R_{uv'}^1(a), \quad R_{uv}^0(b, c) \subset R_{uv'}^1(b, c).$$

- For $1 \leq j \leq m$, let v_j be the only element of $V(T_j) \setminus V(T_{j-1})$, and let w_j be the only element of $V(T_{j-1}) \cap N_T(v_j)$.
- We define $A_j \subset T(r)$ by

$$A_j := B_{v_j}(r) \cup A_{v_j w_j}(r) \cup A_{w_j v_j}(r).$$



Moreover, we define $B_j, C_j \subset Y_{T_j}$ and $D_j, E_j \subset Y_{T_{j-1}}$ by

$$\begin{aligned} B_j &:= R_{v_j w_j}^1(r/2, r] \cup \{(z, t) \mid z \in R_{v_j w_j}^1(r/2), 0 \leq t < d(v_j)\}, \\ C_j &:= (A_j \setminus T_j(r/2)) \cup \{(z, t) \mid z \in A_j \cap \partial T_j(r/2), 0 \leq t < \bar{d}(z)\}, \\ D_j &:= \{(z, t) \mid z \in A_j \cap \partial T_{j-1}(r/2), 0 \leq t < d(w_j)\}, \\ E_j &:= (A_j \setminus T_{j-1}(r/2)) \cup D_j. \end{aligned}$$

Remark 5.13. For each j , $Y_{T_j} \setminus C_j$ is naturally identified with $Y_{T_{j-1}} \setminus E_j$. The identification map $\iota_j : Y_{T_j} \setminus C_j \rightarrow Y_{T_{j-1}} \setminus E_j$ preserves the metric: $\iota_j^* g_{Y_{T_{j-1}}} = g_{Y_{T_j}}$.

Lemma 5.14. For each $j = 2, \dots, m$, there exists a diffeomorphism $\varphi_j : Y_{T_j} \rightarrow Y_{T_{j-1}}$ which satisfies the following properties:

- (a): $\varphi_j|_{Y_{T_j} \setminus C_j} \equiv \iota_j$.
- (b): Any tangent vector ξ on $B_j \cap \varphi_j^{-1}(D_j)$ satisfies $|d\varphi_j(\xi)|_{j-1} \leq |\xi|_j$.
- (c): Any tangent vector ξ on Y_{T_j} satisfies $|d\varphi_j(\xi)|_{j-1} \leq c_5 |\xi|_j$, where c_5 is a positive constant which depends only on n .
- (d): $\varphi_j(R_{v_j w_j}^1[r/2, r]) \subset A_j \setminus B_{w_j}(r)$.

Remark 5.15. In (d), notice that $A_j \setminus B_{w_j}(r) \subset Y_{T_{j-1}}$, since $A_j \setminus B_{w_j}(r)$ is disjoint from $T_{j-1}(r/2)$.

Proof. Let us define $F_j \subset Y_{T_j}$ by

$$F_j := \{(z, t) \mid z \in \partial T_j(r/2) \cap A_j, \bar{d}(z) - d(w_j) \leq t < \bar{d}(z)\}.$$

Then, there exists a diffeomorphism $\bar{\psi}_j : Y_{T_j} \setminus F_j \rightarrow Y_{T_{j-1}} \setminus D_j$ such that

- (i): $\bar{\psi}_j|_{Y_{T_j} \setminus C_j} \equiv \iota_j$.
- (ii): There exists $\delta > 0$ and a diffeomorphism $\psi_j : \partial T_j(r/2) \cap A_j \rightarrow \partial T_{j-1}(r/2) \cap A_j$ such that

$$\bar{\psi}_j(z, \bar{d}(z) - d(w_j) + t) = (\psi_j(z), t) \quad (z \in \partial T_j(r/2) \cap A_j, -\delta \leq t \leq 0).$$

- (iii): Any tangent vector ξ on $R_{v_j w_j}^1(r/2) \subset \partial T_j(r/2) \cap A_j$ satisfies $|d\psi_j(\xi)|_{T(r)} \leq |\xi|_{T(r)}$.

We may assume that $\bar{\psi}_j$ and ψ_j satisfy

- (iv): $\bar{\psi}_j(R_{v_j w_j}^1[r/2, r]) \subset A_j \setminus B_{w_j}(r)$.

(v): Any tangent vector ξ on $Y_{T_j} \setminus F_j$ satisfy $|d\bar{\psi}_j(\xi)|_{j-1} \leq c'_5 |\xi|_j$, where c'_5 is a constant depending only on n .

(iv) can be achieved since $R_{v_j w_j}^1[r/2, r] \cap F_j = \emptyset$. (v) can be achieved since $C_1 \setminus F_1, \dots, C_m \setminus F_m$ and $E_1 \setminus D_1, \dots, E_m \setminus D_m$ are isometric (recall that we have assumed lengths of $F(|e|)$ are same for all $e \in \Sigma_1(T)$ (remark 5.10)).

Finally, we extend $\bar{\psi}_j$ to $\varphi_j : Y_{T_j} \rightarrow Y_{T_{j-1}}$ so that

$$\varphi_j(z, t) := (\psi_j(z), t - \bar{d}(z) + d(w_j)) \quad ((z, t) \in F_j).$$

That φ_j is a diffeomorphism follows from (ii) (we have to check that it is smooth on ∂F_j). Finally, (i) implies (a), (iii) implies (b), (v) implies (c), (iv) implies (d). \square

Take $\varphi_2, \dots, \varphi_m$ as in lemma 5.14, and let $\varphi'' := \varphi_2 \circ \dots \circ \varphi_m : Y_{T_m} \rightarrow Y_{T_1}$.

Lemma 5.16. *Any tangent vector ξ on Y_{T_1} satisfies $|d\varphi''(\xi)|_1 \leq (c_5)^3 \cdot |\xi|_m$.*

Proof. For any $y \in Y_{T_m}$ and $1 \leq j \leq m$, define $y_j \in Y_{T_j}$ by $y_m := y$, $y_{j-1} := \varphi_j(y_j)$. We prove the following claim: Let $2 \leq j \leq m$. If there exists $\xi \in T_{y_j} Y_{T_j}$ such that $|d\varphi_j(\xi)|_{j-1} > |\xi|_j$, then at least one of the followings holds:

- (A): Any $k > j$ and $\zeta \in T_{y_k} Y_{T_k}$ satisfies $|d\varphi_k(\zeta)|_{k-1} = |\zeta|_k$.
- (B): The number of $k < j$, such that there exists $\zeta \in T_{y_k} Y_{T_k}$ satisfying $|d\varphi_k(\zeta)|_{k-1} > |\zeta|_k$, is at most 1.

The above claim implies that the number of j such that there exists $\xi \in T_{y_j} Y_{T_j}$ satisfying $|d\varphi_j(\xi)|_{j+1} > |\xi|_j$ is at most 3. Hence lemma 5.14(c) implies lemma 5.16.

We prove the above claim. Assume that there exists $\xi \in T_{y_j} Y_{T_j}$ such that $|d\varphi_j(\xi)|_{j-1} > |\xi|_j$. (a) and remark 5.13 imply that $y_j \in C_j$, hence $y_{j-1} \in E_j$. We consider the following two cases:

- (i): There exists $u \in N_T(v_j) \setminus \{w_j\}$ such that $y_j \in R_{v_j u}^0(r/2, r] \cup \{(z, t) \mid z \in R_{v_j u}^0(r/2), 0 \leq t < d(v_j)\}$.
- (ii): Otherwise.

First we consider the case (i). By remark 5.12, $y_j \in B_j$. Notice that E_j is divided into three parts:

$$D_j, \quad A_j \setminus B_{w_j}(r), \quad R_{w_j v_j}^0(r/2, r].$$

- (i)-(i): Assume that $y_{j-1} \in D_j$. Then, $y_j \in B_j \cap \varphi_j^{-1}(D_j)$. Hence (b) implies $|d\varphi_j(\xi)|_{j-1} \leq |\xi|_j$ for any $\xi \in T_{y_j} Y_{T_j}$, contradicting the assumption.
- (i)-(ii): Assume that $y_{j-1} \in A_j \setminus B_{w_j}(r)$. In this case, $y_k \notin C_k$ for any $k < j$. Hence (a) and remark 5.13 imply that $|d\varphi_k(\zeta)|_{k-1} = |\zeta|_k$ for any $k < j$ and $\zeta \in T_{y_k} Y_{T_k}$. Therefore (B) holds.
- (i)-(iii): Assume that $y_{j-1} \in R_{w_j v_j}^0(r/2, r]$. Let k_0 be the unique integer such that $w_j = v_{k_0}$. We claim that $|d\varphi_k(\zeta)|_{k-1} = |\zeta|_k$ for any $k < j$, $k \neq k_0$ and $\zeta \in T_{y_k} Y_{T_k}$ (hence (B) holds). This is proved as follows. When $k_0 < k < j$, $y_k \notin C_k$. Hence the claim

follows from (a) and remark 5.13. Moreover, (d) implies that $y_{k_0-1} \in A_{k_0} \setminus B_{w_{k_0}}(r)$. Hence (i)-(ii) proves the claim for $k < k_0$.

We consider the case (ii). In this case, $y_k \notin C_k$ for any $k > j$. Hence (a) and remark 5.13 imply that $|d\varphi_k(\zeta)|_{k-1} = |\zeta|_k$ for any $k > j$, $\zeta \in T_{y_k} Y_{T_k}$. Hence (A) holds. \square

Finally we prove lemma 5.11. Since $T_1 = \{v_1\}$, $Y_{T_1} = T(r) \setminus B_{v_1}(r/2)$. Hence there exists a diffeomorphism $\varphi''' : Y_{T_1} \rightarrow T(r) \setminus \{i(v_1)\}$ such that $|d\varphi'''(\xi)|_{T(r)}/|\xi|_1 \leq \text{const}_n$ for any nonzero tangent vector ξ on Y_{T_1} .

Hence $\varphi' := \varphi''' \circ \varphi''$ satisfies $|d\varphi'(\xi)|_{T(r)}/|\xi|_{X_T} = |d\varphi'(\xi)|_{T(r)}/|\xi|_m \leq \text{const}_n$ for any nonzero tangent vector ξ on $Y = Y_{T_m}$.

6. THICK TRIANGULATION: PROOF

This section is devoted to the proof of lemma 3.3. Our idea to prove lemma 3.3 is to use the notion of *Delaunay triangulation*.

In section 6.1, we introduce the notion of Delaunay triangulation on the Euclidean space. Although it seems well-known, we need a result which is suitable for our applications (theorem 6.1). After some preparations in section 6.2, we define the notion of Delaunay triangulation on Riemannian manifolds in section 6.3 (theorem 6.5), and prove lemma 3.3 as an application.

6.1. Delaunay triangulation on the Euclidean space. First we introduce some conditions for subsets of \mathbb{R}^n . Let S be a subset of \mathbb{R}^n , and a, b, c, d be positive real numbers. We define conditions $P_1(a)$, $P_2(b)$, $P_3(c)$, $P_4(d)$ for S as follows:

- S satisfies $P_1(a) \iff$ Any $s, t \in S, s \neq t$ satisfy $\text{dist}(s, t) \geq a$.
- S satisfies $P_2(b) \iff$ Any $x \in \mathbb{R}^n$ satisfies $B^n(x, b) \cap S \neq \emptyset$.
- S satisfies $P_3(c) \iff$ Any $x \in \mathbb{R}^n$ and $0 < r < c$ satisfy $|S^{n-1}(x, r) \cap S| \leq n + 1$.
- S satisfies $P_4(d) \iff$ Any $s_0, \dots, s_n \in S$ satisfying $\text{diam}(s_0, \dots, s_n) \leq d$ is nondegenerate (i.e. there exists no hyperplane in \mathbb{R}^n which contains s_0, \dots, s_n).

For any $s \in S$, define $V(s) \subset \mathbb{R}^n$ by $V(s) := \{x \in \mathbb{R}^n \mid \text{dist}(x, s) = \text{dist}(x, S)\}$, and define a symplcial complex X_S by $V(X_S) = S$,

$$\Sigma(X_S) := \{\{s_0, \dots, s_k\} \subset S \mid V(s_0) \cap \dots \cap V(s_k) \neq \emptyset\}.$$

Moreover, define $F_S : |X_S| \rightarrow \mathbb{R}^n$ so that $F_S|_{|\sigma|}$ is an affine map for any $\sigma \in \Sigma(X_S)$.

Theorem 6.1. *If $S \subset \mathbb{R}^n$ satisfies $P_1(a)$, $P_2(b)$, $P_3(c)$, $P_4(d)$ and $c > b$, $d > 2b$, then (X_S, F_S) is a triangulation of \mathbb{R}^n .*

Proof. By remark 3.2, it is enough to show that F_S is nondegenerate and homeomorphism. First we show the nondegeneracy. If $s, t \in S$ satisfies $V(s) \cap V(t) \neq \emptyset$, then $P_2(b)$ implies that $\text{dist}(s, t) \leq 2b$. Hence any $\sigma \in \Sigma_n(X_S)$ satisfies $\text{diam}(\sigma) \leq 2b$. Since $d > 2b$ and S satisfies $P_4(d)$, σ is nondegenerate, hence $F_S|_{|\sigma|}$ is nondegenerate.

We have to show that F_S is a homeomorphism. Since F_S is clearly continuous, it is enough to show its properness, injectivity, and surjectivity.

We show the properness of F_S . Let K be a compact set on \mathbb{R}^n , and assume that $\sigma \in \Sigma_n(X_S)$ satisfies $F_S(|\sigma|) \cap K \neq \emptyset$. Then, $\text{diam}(\sigma) \leq 2b$ implies that $\sigma \subset B(K, 2b) := \bigcup_{x \in K} B^n(x, 2b)$. On the otherhand, since S satisfies $P_1(a)$, $S \cap B(K, 2b)$ is finite. Hence $F_S^{-1}(K)$ is contained in a compact set $\bigcup_{\sigma \subset B(K, 2b)} |\sigma|$.

We show the injectivity of F_S . Arguing indirectly, suppose that there exist $x, y \in |X_S|$ such that $x \neq y$ and $F_S(x) = F_S(y)$. Take $\sigma, \tau \in \Sigma_n(X_S)$ such that $x \in |\sigma|$, $y \in |\tau|$. Since $F_S|_{|\sigma|}$, $F_S|_{|\tau|}$ are injective, $\sigma \neq \tau$. Define $p, q \in \mathbb{R}^n$ and $d_p, d_q > 0$ by $\{p\} := \bigcap_{s \in \sigma} V(s)$, $d_p := \text{dist}(p, S)$, $\{q\} := \bigcap_{t \in \tau} V(t)$, $d_q := \text{dist}(q, S)$. Since S satisfies $P_2(b)$, $d_p, d_q \leq b$. Then, since $c > b$ and S satisfies $P_3(c)$, $\sigma = S \cap S^{n-1}(p, d_p)$, $\tau = S \cap S^{n-1}(q, d_q)$.

If $S^{n-1}(p, d_p) \cap S^{n-1}(q, d_q) = \emptyset$, $F_S(|\sigma|) \cap F_S(|\tau|) = \emptyset$: a contradiction. If $S^{n-1}(p, d_p) \cap S^{n-1}(q, d_q) \neq \emptyset$, one can easily shows that $x, y \in |\sigma \cap \tau|$. However it contradicts the fact that $F_S|_{|\sigma|}$, $F_S|_{|\tau|}$ are injective.

Finally we show the surjectivity, i.e. $F_S(|X_S|) = \mathbb{R}^n$. First we claim that for any $x \in \bigcup_{\dim \sigma \geq n-1} |\sigma|$, $F_S(|X_S|)$ is a neighborhood of $F_S(x)$. Let σ be a unique simplex of X_S such that $x \in \text{int}|\sigma|$. If $\sigma \in \Sigma_n(X_S)$, then the claim is obvious since $F_S|_{|\sigma|}$ is nondegenerate. If $\sigma \in \Sigma_{n-1}(X_S)$, there exists a unique hyperplane $\pi \subset \mathbb{R}^n$, which contains σ . π divides \mathbb{R}^n into two halfspaces H_1, H_2 , and there exist $s_1 \in H_1 \cap S$, $s_2 \in H_2 \cap S$ such that $\sigma \cup \{s_1\}, \sigma \cup \{s_2\} \in \Sigma_n(X_S)$. Then $F_S(|\sigma \cup \{s_1\}| \cup |\sigma \cup \{s_2\}|)$ is a neighborhood of $F_S(x)$.

Assume that $F_S(|X_S|) \subsetneq \mathbb{R}^n$, and take $x \in \mathbb{R}^n \setminus F_S(|X_S|)$. Then, there exists $e \in S^{n-1}$ such that $(x + e\mathbb{R}) \cap F_S(|X_S|) \neq \emptyset$, and $(x + e\mathbb{R}) \cap \bigcup_{\dim \sigma \leq n-2} F_S(|\sigma|) = \emptyset$. Then, the above claim shows that $T = \{t \in \mathbb{R} \mid x + et \in F_S(|X_S|)\}$ is open in \mathbb{R} . Moreover, the properness of F_S shows that T is closed. Finally, obviously $T \neq \emptyset$, $0 \notin T$: a contradiction. \square

6.2. Nondegenerate conditions. Our idea to prove lemma 3.3 is to generalize the notion of Delaunay triangulation for finite sets on Riemannian manifolds. To carry out this argument, we consider finite sets which satisfy some nondegenerate conditions. In this subsection, we spell out those nondegenerate conditions, and show the existence of a finite set satisfying those conditions (lemma 6.2).

First we introduce some notations:

- For $x \in N$ and $0 \leq a < b$, $B_N(x : a, b) := \{y \in N \mid \text{dist}_N(x, y) \in [a, b]\}$.
- For $k = 1, \dots, n$ and $x_0, \dots, x_k \in N$ such that $\text{diam}(x_0, \dots, x_k) \leq \text{inj}(N)$, we define

$$\theta_k(x_0, \dots, x_k) := \inf_{\sigma} \text{vol}(e_{x_{\sigma(0)}x_{\sigma(1)}}, \dots, e_{x_{\sigma(0)}x_{\sigma(k)}}),$$

where σ runs over all permutations of $\{0, \dots, k\}$, and

$$\text{vol}(v_1, \dots, v_k) := \sqrt{\det(v_i \cdot v_j)_{1 \leq i, j \leq k}}.$$

We introduce some conditions for finite sets on N . Let S be a finite subset of N , and $a, b, c, d, \delta, \theta$ are positive numbers, and $k = 2, \dots, n$:

- S satisfies $P'_1(a) \iff$ Any $s, t \in S$, $s \neq t$ satisfy $\text{dist}_N(s, t) \geq a$.
- S satisfies $P'_2(b) \iff$ Any $x \in N$ satisfies $B_N(x, b) \cap S \neq \emptyset$.
- S satisfies $P'_3(c, \delta) \iff$ Any $x \in N$ and $0 < r \leq c$ satisfy $|S \cap B_N(x : r, (1+\delta)r)| \leq n+1$.
- S satisfies $P'_4(k, d, \theta) \iff$ Any $s_0, \dots, s_k \in S$ satisfying $\text{diam}(s_0, \dots, s_k) \leq d$ satisfies $\theta_k(s_0, \dots, s_k) \geq \theta$.

Lemma 6.2. *There exist positive numbers δ, θ such that: if $\varepsilon > 0$ is sufficiently small, then there exists a finite set $S \subset N$ which satisfies $P'_1(\varepsilon/2)$, $P'_2(2\varepsilon)$, $P'_3(10\varepsilon, \delta)$, $P'_4(n, 10\varepsilon, \theta)$.*

Remark 6.3. We abbreviate $P'_1(\varepsilon/2) \wedge P'_2(2\varepsilon) \wedge P'_3(10\varepsilon, \delta) \wedge P'_4(n, 10\varepsilon, \theta)$ as $P'(\varepsilon, \delta, \theta)$.

Proof. We prove the following slightly stronger result:

There exist positive numbers $\delta, \theta_2, \dots, \theta_n$ such that: if $\varepsilon > 0$ is sufficiently small, then there exists a finite set $S \subset N$ which satisfies $P'_1(\varepsilon/2)$, $P'_2(2\varepsilon)$, $P'_3(10\varepsilon, \delta)$, $P'_4(k, 10\varepsilon, \theta_k)$ ($k = 2, \dots, n$).

First note that if $S \subset N$ is a maximal set which satisfies $P'_1(\varepsilon)$, then it also satisfies $P'_2(\varepsilon)$. Hence, for any $\varepsilon > 0$, there exists $S \subset N$ which satisfies $P'_1(\varepsilon)$, $P'_2(\varepsilon)$.

Take $S = \{s_1, \dots, s_m\} \subset N$ so that it satisfies $P'_1(\varepsilon)$, $P'_2(\varepsilon)$. Then, any $S' = \{s'_1, \dots, s'_m\} \subset N$ such that $s'_i \in B(s_i, \varepsilon/10)$ ($i = 1, \dots, m$) satisfies $P'_1(4\varepsilon/5)$, $P'_2(11\varepsilon/10)$ (hence it obviously satisfies $P'_1(\varepsilon/2)$, $P'_2(2\varepsilon)$).

We show that there exist $\delta, \theta_2, \dots, \theta_n$, such that for sufficiently small $\varepsilon > 0$ we can take S' so that it satisfies $P'_3(10\varepsilon, \delta)$, $P'_4(2, 10\varepsilon, \theta_2), \dots, P'_4(n, 10\varepsilon, \theta_n)$.

We construct S' inductively. First, let $s'_1 := s_1$. Suppose that we have chosen s'_1, \dots, s'_l so that $\{s'_1, \dots, s'_l\}$ satisfies $P'_3(10\varepsilon, \delta)$, $P'_4(2, 10\varepsilon, \theta_2), \dots, P'_4(n, 10\varepsilon, \theta_n)$.

We define $W_2, \dots, W_n, Z \subset B_N(s_{l+1}, \varepsilon/10)$ as follows:

- W_k is the set of $s \in B_N(s_{l+1}, \varepsilon/10)$ such that $\{s'_1, \dots, s'_l, s\}$ does not satisfy $P'_4(k, 10\varepsilon, \theta_k)$.
- Z is the set of $x \in B_N(s_{l+1}, \varepsilon/10)$ such that $\{s'_1, \dots, s'_l, x\}$ does not satisfy $P'_3(10\varepsilon, \delta)$.

Then, the volume of W_2, \dots, W_n, Z are estimated as follows:

- For any $c > 0$, there exists $\Theta_2 > 0$ (depending on c) and $E > 0$ (depending on c, Θ_2, N) such that: if $\theta_2 \leq \Theta_2$ and $\varepsilon \leq E$, then $\text{vol}(W_2)/\text{vol}(B_N(s_{l+1}, \varepsilon/10)) \leq c$.
- For any $c > 0$ and $k = 3, \dots, n$, there exists $\Theta_k > 0$ (depending on c, θ_{k-1}) and $E' > 0$ (depending on $c, \theta_{k-1}, \Theta_k, N$) such that: if $\theta_k \leq \Theta_k$ and $\varepsilon \leq E'$, then $\text{vol}(W_k)/\text{vol}(B_N(s_{l+1}, \varepsilon/10)) \leq c$.

- For any $c > 0$, there exists $\Delta > 0$ (depending on c, θ_n) and $E'' > 0$ (depending on c, θ_n, Δ, N) such that: if $\delta \leq \Delta$ and $\varepsilon \leq E''$, then $\text{vol}(Z)/\text{vol}(B_N(s_{l+1}, \varepsilon/10)) \leq c$.

Therefore, when $\theta_2, \dots, \theta_n, \delta$ are properly chosen and $\varepsilon > 0$ is sufficiently small, then $\text{vol}(W_2 \cup \dots \cup W_n \cup Z)/\text{vol}(B(s_{l+1}, \varepsilon/10)) < 1$. Therefore we can take $s'_{l+1} \in B_N(s_{l+1}, \varepsilon/10)$ such that $\{s'_1, \dots, s'_{l+1}\}$ satisfies $P'_3(10\varepsilon, \delta), P'_4(2, 10\varepsilon, \theta_2), \dots, P'_4(n, 10\varepsilon, \theta_n)$. Continuing this process until $l+1 = m$, we can construct S' which satisfies $P'_3(10\varepsilon, \delta), P'_4(k, 10\varepsilon, \theta_k)$ ($k = 2, \dots, n$). \square

6.3. Delaunay triangulations on Riemannian manifolds. In this subsection, we define the notion of Delaunay triangulation for finite sets on Riemannian manifolds, and prove lemma 3.3 as an application.

Let $\varepsilon, \delta, \theta$ be positive numbers as in lemma 6.2, N be a closed Riemannian manifold, and S be a finite set on N which satisfies $P'(\varepsilon, \delta, \theta)$. First we construct a symplcial complex X_S and a smooth map $F_S : |X_S| \rightarrow N$, and then prove that (X_S, F_S) is a triangulation of N when $\varepsilon > 0$ is sufficiently small (theorem 6.5).

X_S is defined in the exactly same way as in the case of the Euclidean space (section 6.1). For each $s \in S$, we define $V(s) \subset N$ by

$$V(s) := \{x \in N \mid \text{dist}_N(s, x) = \text{dist}_N(S, x)\}.$$

Then, we define a symplcial complex X_S by $V(X_S) = S$, and

$$\Sigma(X_S) := \{\{s_0, \dots, s_k\} \subset S \mid V(s_0) \cap \dots \cap V(s_k) \neq \emptyset\}.$$

Since S satisfies $P'_2(2\varepsilon)$, any $\sigma \in \Sigma(X_S)$ satisfies $\text{diam}(\sigma) \leq 4\varepsilon$.

Next we define $F_S : |X_S| \rightarrow N$. The definition consists of 3 steps.

Step1: For any $s \in S$, there exists an unique map $i_s : \overline{\text{St}}_{X_S}(s) \rightarrow T_s N$ such that

- $i_s(t) = \overrightarrow{st}$ for any $t \in N_{X_S}(s)$.
- For any $\sigma \in \Sigma(X_S)$ such that $s \in \sigma$, $i_s|_{|\sigma|}$ is an affine map.

We define $F_s : \overline{\text{St}}_{X_S}(s) \rightarrow N$ by $F_s := \exp_s \circ i_s$.

Step2: For each $k \geq 1$ we define

$$\mu_k : \{(q_0, \dots, q_k) \in N^{k+1} \mid \text{diam}\{q_0, \dots, q_k\} < \text{inj}(N)\} \times \Delta^k \rightarrow N,$$

where

$$\Delta^k := \{(t_0, \dots, t_k) \mid t_0, \dots, t_k \geq 0, t_0 + \dots + t_k = 1\}.$$

Take an arbitrary function $\rho : [0, 1] \rightarrow [0, 1]$, such that $\rho \equiv 0$ on some neighborhood of 0, and $\rho \equiv 1$ on some neighborhood of 1.

When $k = 1$, we define μ_1 by

$$\mu_1(q_0, q_1, (t_0, t_1)) := \gamma_{q_0, q_1}(\rho(t_0)).$$

Suppose that we have defined μ_1, \dots, μ_{k-1} . Then, we define μ_k by

$$\mu_k(q_0, q_1, \dots, q_k, (t_0, \dots, t_k)) := \begin{cases} q_k & (t_0 = \dots = t_{k-1} = 0, t_k = 1) \\ \gamma_{\mu_{k-1}(q_0, \dots, q_{k-1}, (t_0/1-t_k, \dots, t_{k-1}/1-t_k)), q_k}(\rho(t_k)) & (\text{otherwise}) \end{cases}.$$

As is clear from the definition, μ_k is C^∞ .

Step3: Fix an arbitrary order on S . Then, for any $\sigma = \{s_0, \dots, s_k\} \in \Sigma(S)$, where $s_0 < \dots < s_k$, we define $F_S|_{|\sigma|}$ by

$$F_S(x) := \mu_k(F_{s_0}(x), \dots, F_{s_k}(x), (t_0, \dots, t_k)) \quad (x = s_0 t_0 + \dots + s_k t_k, t_0 + \dots + t_k = 1).$$

This completes the definition of $F_S : |X_S| \rightarrow N$.

We need the following lemma for later arguments:

Lemma 6.4. (1) $F_S|_{|\sigma|}$ is a geodesic for any $\sigma \in \Sigma_1(X_S)$.
(2) When ε is sufficiently small, $F_S(|\sigma|) \subset B_N(\sigma, 8\varepsilon)$ for any $\sigma \in \Sigma(X_S)$.

Proof. (1): Set $\sigma := \{p, q\}$. Then, for $x = \lambda p + (1 - \lambda)q \in |\sigma|$, $F_p(x) = F_q(x) = \gamma_{pq}(\lambda)$. Therefore $F_S(\lambda p + (1 - \lambda)q) = \gamma_{pq}(\lambda)$, hence $F_S|_{|\sigma|}$ is a geodesic.

(2): Since $\text{diam}(\sigma) \leq 4\varepsilon$, $F_{s_i}(|\sigma|) \subset B_N(s_i, 4\varepsilon)$ for each $i = 0, \dots, k$. Therefore $F_{s_i}(|\sigma|) \subset B_N(s_0, 8\varepsilon)$ for each i . On the otherhand, when ε is sufficiently small, $B_N(x, 8\varepsilon)$ is geodesically convex for any $x \in N$. Hence $F_S(|\sigma|) \subset B_N(s_0, 8\varepsilon) \subset B_N(\sigma, 8\varepsilon)$. \square

We prove that when ε is sufficiently small, (X_S, F_S) is a triangulation of N :

Theorem 6.5. Let δ, θ be positive numbers as in lemma 6.2, and N be a closed Riemannian manifold. Then, for sufficiently small $\varepsilon > 0$, the following holds:

If $S \subset N$ satisfies $P'(\varepsilon, \delta, \theta)$, (X_S, F_S) is a triangulation of N .

The proof is based on the next lemma 6.7. First we need the following definition:

Definition 6.6. Let $(X_i)_{i=1,2,\dots}$ be a sequence of subsets of \mathbb{R}^n , and X_∞ be a subset of \mathbb{R}^n .

- (1) When X_∞ is a finite set, $(X_i)_i$ converges to X_∞ if and only if $\sharp X_i = \sharp X_\infty (= m)$ for sufficiently large i , and $(X_i)_i$ converges to X_∞ as elements of

$$\{(x_1, \dots, x_m) \in (\mathbb{R}^n)^m \mid i \neq j \implies x_i \neq x_j\} / \mathfrak{S}_m.$$

- (2) When $X_\infty \cap B^n(r)$ is a finite set for any $r > 0$, $(X_i)_i$ converges to X_∞ if and only if there exists an increasing sequence $(r_j)_j$ of positive real numbers such that $\lim_{j \rightarrow \infty} r_j = \infty$ and $(X_i \cap B^n(r_j))_i$ converges to $X_\infty \cap B^n(r_j)$ for any j .

Lemma 6.7. Let $(\varepsilon_i)_i$ be a sequence of positive numbers, $(S_i)_i$ be a sequence of finite sets on N , such that $\lim_{i \rightarrow \infty} \varepsilon_i = 0$, and each S_i satisfies $P'(\varepsilon_i, \delta, \theta)$. Let $(p_i)_i$ be a sequence of points on N , and (U_i, φ_i) be a local chart on N which is defined near p_i , and satisfies the following conditions:

- $\varphi_i(U_i) = \{(x_1, \dots, x_n) \mid |x| < \text{inj}(N)\}$.
- $\varphi_i(p_i) = (0, \dots, 0)$.
- (x_1, \dots, x_n) is a geodesic coordinate on N .

Let us define $T_i \subset \mathbb{R}^n$ by $T_i := \varphi_i(U_i \cap S_i) / \varepsilon_i$. Then, the following holds:

- (1) Up to subsequence, $(T_i)_i$ converges to some $T_\infty \subset \mathbb{R}^n$ in the sense of definition 6.6.
- (2) For any $a < 1/2$, $b > 2$, $c, d < 10$, T_∞ satisfies $P_1(a)$, $P_2(b)$, $P_3(c)$, $P_4(d)$.

Proof. Fix arbitrary $a < 1/2$. Then, for any $r > 0$, $T_i \cap B^n(r)$ satisfies $P_1(a)$ for sufficiently large i . Hence $\sharp(T_i \cap B^n(r))$ is bounded uniformly on i . Therefore, up to subsequence $(T_i \cap B^n(r))_i$ converges to a certain finite subset of $B^n(r)$. Then, the diagonal argument proves (1). (2) is an immediate consequence of the assumption that S_i satisfies $P'(\varepsilon_i, \delta, \theta)$ for each i . \square

Now we can prove theorem 6.5. By remark 3.2, it is enough to show that F_S is nondegenerate and homeomorphic. Moreover, since F_S is continuous and $|X_S|$ is compact, it is enough to show the following two assertions:

- (1) When $\varepsilon > 0$ is sufficiently small, F_S is nondegenerate.
- (2) When $\varepsilon > 0$ is sufficiently small, F_S is a bijection.

(1) follows from theorem 6.1, using lemma 6.7 (notice that one can take b, c, d so that $b > 2$, $b < c < 10$, $2b < d < 10$).

We prove (2). For any $s \in S$, we define $X_s \subset X_S$ by

$$V(X_s) := S \cap B_N(s, 100\varepsilon), \quad \Sigma(X_s) := \Sigma(X_S) \cap 2^{V(X_s)}.$$

Notice that the following two assertions follow from theorem 6.1, using lemma 6.7:

- (2'): When $\varepsilon > 0$ is sufficiently small, $F_S|_{|X_s|}$ is injective for any $s \in S$.
- (2''): When $\varepsilon > 0$ is sufficiently small, $B_N(s, 2\varepsilon) \subset F_S(|X_s|)$ for any $s \in S$.

Finally we show that (2) follows from (2') and (2''). Suppose that $\varepsilon > 0$ is so small that $F_S|_{|X_s|}$ is injective, and $B_N(s, 2\varepsilon) \subset F_S(|X_s|)$ for any $s \in S$.

If F_S is not injective, there exist $x, y \in |X_S|$ such that $x \neq y$ and $F_S(x) = F_S(y)$. Take $\sigma, \tau \in \Sigma(X_S)$ such that $x \in |\sigma|$, $y \in |\tau|$. Since $F_S(|\sigma|) \cap F_S(|\tau|) \neq \emptyset$, lemma 6.4 (2) implies that $\tau \subset B_N(s, 100\varepsilon)$ for any $s \in \sigma$. However it is a contradiction, since $F_S|_{X_s}$ is injective for any $s \in S$.

On the otherhand, since S satisfies $P'_2(2\varepsilon)$, $\bigcup_{s \in S} B_N(s, 2\varepsilon) = N$. Hence F_S is surjective.

This completes the proof of theorem 6.5.

Finally, we prove lemma 3.3. Take δ, θ as in lemma 6.2. Let $\varepsilon > 0$ be a sufficiently small positive number, S be a finite set on N satisfying $P'(\varepsilon, \delta, \theta)$, and (X_S, F_S) be the triangulation, which is defined as above. We claim that (X_S, F_S) satisfies (1), (2), (3) in lemma 3.3 when $\varepsilon > 0$ is sufficiently small. (1) was confirmed in lemma 6.4 (1). (2) follows from that any $\sigma := \{s_0, \dots, s_n\} \in \Sigma_n(X_S)$ satisfies $\text{dist}_N(s_i, s_j) \in [\varepsilon/2, 4\varepsilon]$ and $\theta(s_0, \dots, s_n) \geq \theta$. (3) follows from that any $\sigma = \{s_0, s_1\} \in \Sigma_1(X_S)$ satisfies $\text{dist}_N(s_0, s_1) \leq 4\varepsilon$ and S satisfies $P'_1(\varepsilon/2)$.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY, KYOTO 606-8502,
JAPAN

E-mail address: iriek@math.kyoto-u.ac.jp